A Project in Mathematics

The Crystallographic Restriction Theorem in the Euclidean Plane

Its Proof and Meaning

Written by:	Boris Figovsky
	"Leo-Baeck" High School, Haifa, Israel
Adult Sponsor:	Dan Guralnik (The Technion, Haifa, Israel)
	The Faculty of Mathematics.

Preface

The purpose of this project is to understand the **crystallographic restriction theorem** -

> There exists exactly 17 isomorphism-types of tessellation groups in the Euclidean plane.

The main purpose of this project, except presenting a proof for this theorem in its general form, is to understand the mathematical ideas that made it possible to express, from a synthetic point of view, terms as **'tile'** and **'tessellation'**, which look like 'trivial' when we look at the following drawings:

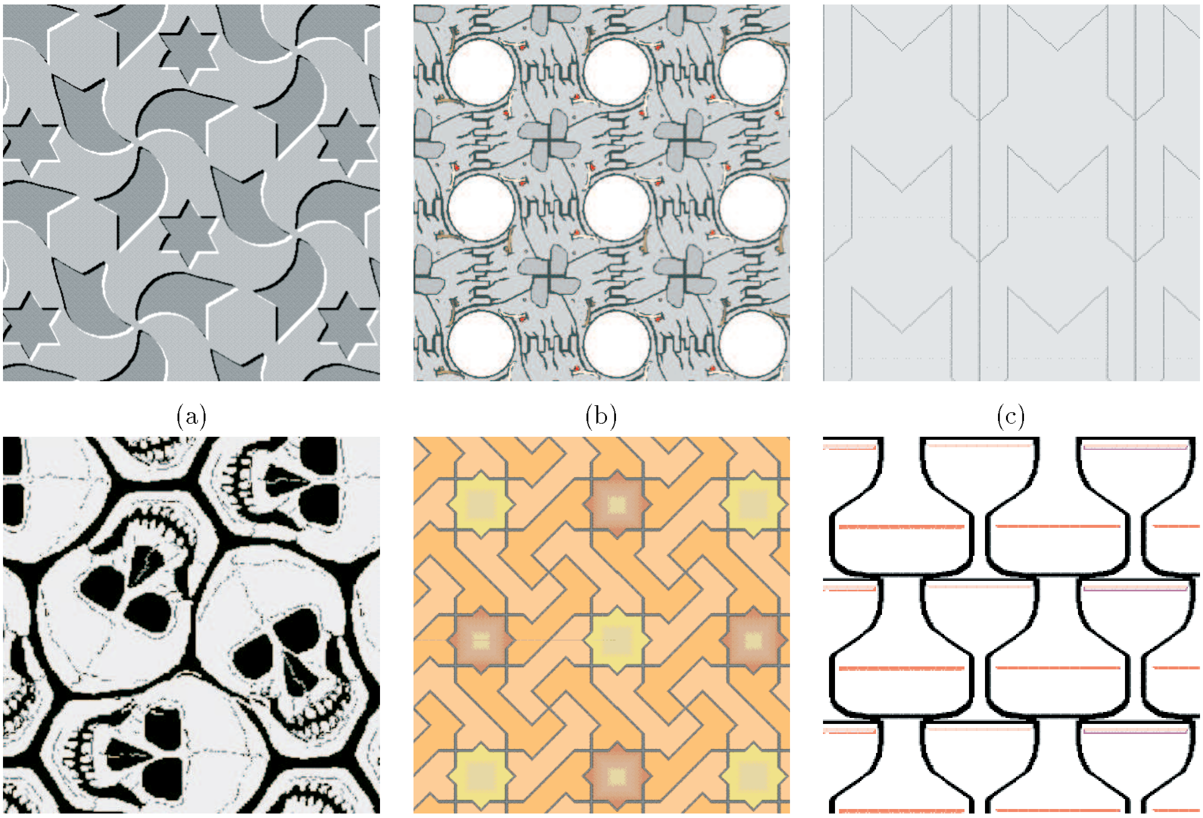


Figure 1: Examples of Tessellations

By looking at the drawing, we get an initial understanding - but not a formal definition - of the term **'tessellation'**. In each of the drawings, it is clear what **'tile'** we need to put on the plane **in a specific order** (with infinitely many copies) to fill the whole plane. The problem is the understanding of the mathematical meaning of the term **'tile'**, and the meaning to be referred to **'a specific order'**.

The solution to this problem was given in the 19^{th} century, while using the terms **'lattice'**, **'isometry'** and **'symmetry group**. If we look again at the drawings, we can see the common symmetries in each pair of drawings denoted by the same letter in figure 2 on the next page.

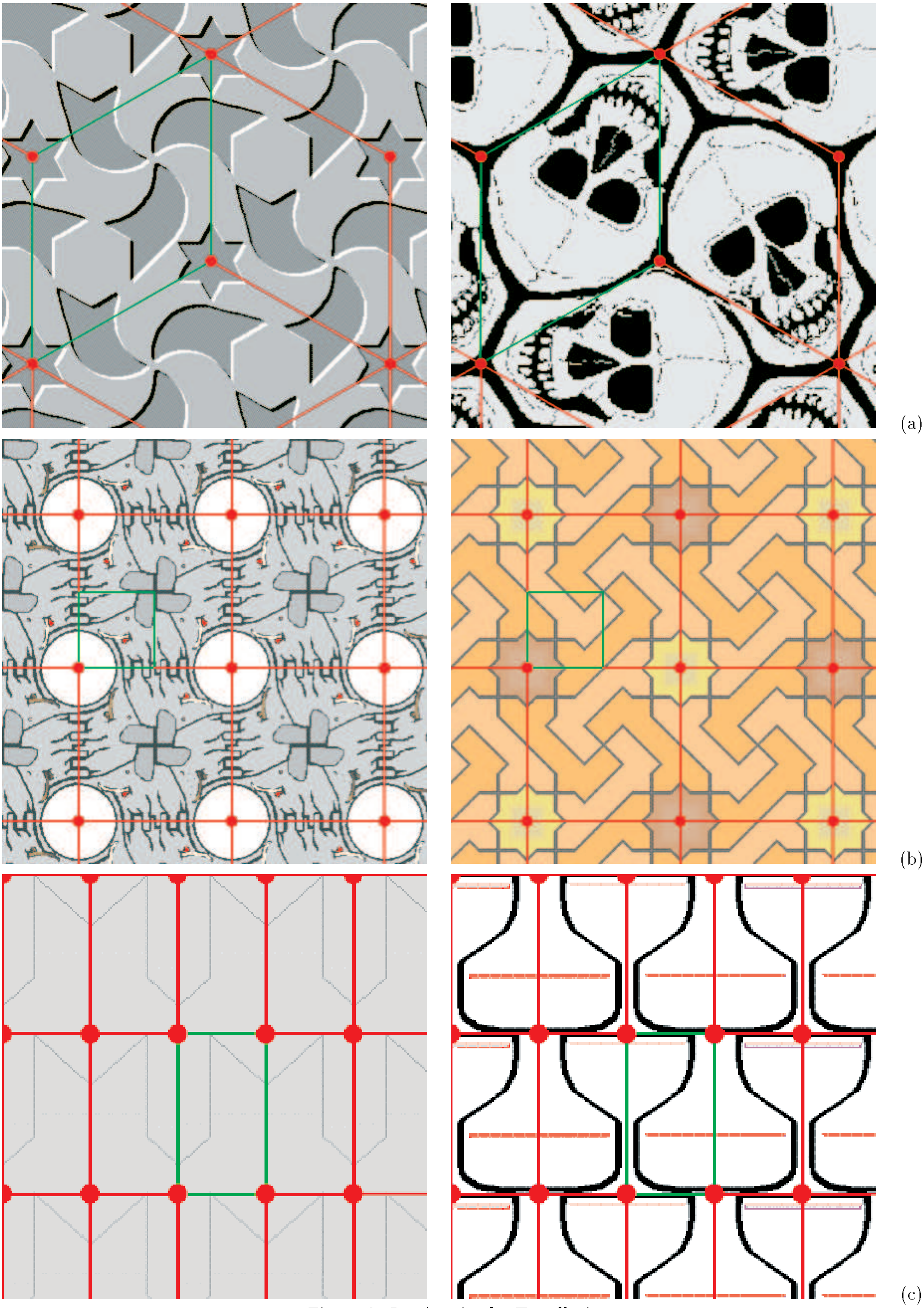


Figure 2: Lattices in the Tessellations

From these drawings, we see that we can replace the original tile with a standard one (triangle or parallelogram), and the whole tiling we can get by translations, rotations or reflections of the standard tile by the same axises and the same centers of the same isometries that copied the original (natural) tile.
This observation allowed the first complete account of the enumeration of the isomorphism-types of tessellation groups on the Euclidean plane and space, spherical geometry, hyperbolic plane and space, and higher dimensions. This account was based on the symmetry-group of lattices in these spaces. The development of topology during the 20^{th} century allowed the final mathematical definition for all the terms mentioned earlier, and phenomena as **crystallographic restriction theorem** has even got a name - **toughness theorems**.
Our goal will be reached when we are able to present the connections between lattices on the plane and their symmetry-groups.
The first chapter introduces the mathematical analysis of tessellation groups. It starts with lattices, and Barlow's proof of the first form of the Theorem. Then, general tessellation groups are defined, and some basic properties are mentioned. The next step is replacing the tile with a standard one, which is based on the group itself. And, the final topic is symmetry-group of a lattice as a tessellation group.
The second chapter presents a full proof of the Theorem. The first section shows the connection between lattices and their symmetry-groups. The second section presents another proof of the first form of the Theorem, and the third and last section finished the proof of the Theorem.

Contents

List of Figures

Chapter 1

Crystallography in the Euclidean Place

This chapter is dealing with the theoretical base of crystallography in the Euclidean place.

1.1 Lattices

Lattices and their symmetry-groups will be a easy-to-use tool to prove the Crystallographic Restriction Theorem.

Definition 1.1.1 (Lattice) *Given a point $O \in \mathbb{E}^2$ and two translations $T_1, T_2 \in \mathrm{Isom}(\mathbb{E}^2)$ such that $T_2(O) \notin OT_1(O)$. The set $L \triangleq \langle T_1, T_2 \rangle \cdot O$ is called a* ***lattice****. The group $\langle T_1, T_2 \rangle$ is denoted by $T(L)$.*

It can be easily verified that $T(L)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ by $T_1^m T_2^n \mapsto (m, n)$. Thus, $T(L)$ is countable. We are interested in the symmetry-group of a lattice: we shall call such group a **Lattice-Group** $\mathrm{Sym}(L)$. It is trivial that $T(L) \subseteq \mathrm{Sym}(L)$.

Definition 1.1.2 (Direct Subgroup) *Given $G < \mathrm{Isom}(\mathbb{E}^2)$, G_+ is the group of all direct isometries in G.*

The direct lattice-group contains half-turns:

Preposition 1.1.1 *Given a lattice L and $O \in L$, then $\langle H_O \rangle \ltimes T(L) \subseteq \mathrm{Sym}_+(L)$*

Proof: Given $X \in L$, we need to represent $H_O(X)$ as $T'(O)$ for some $T' \in T(L)$. If $X = T(O)$ and $X' = T^{-1}(O)$ for some $T \in T(L)$, then $O = T^0(O)$ is the midpoint of $\overline{XX'}$, therefore $H_O(X) = X'$ and $H_O \in \mathrm{Sym}_+(L)$. Finally, it is trivial that $T(L) < \mathrm{Sym}_+(L)$: if $f \in \mathrm{Sym}_+(L)$, then ${}^fT \in \mathrm{Sym}_+(L)$, but ${}^fT$ is a translation. ■
It is worth mentioning the following for any symmetry group:

Lemma 1.1.2 *Given a non-empty set $A \subseteq \mathbb{E}^2$, then $[\mathrm{Sym}(A) : \mathrm{Sym}_+(A)] | 2$.*

Proof: Let's define: $N = \mathrm{Isom}_+(\mathbb{E}^2), H = \mathrm{Sym}(A)$, then $H \cap N = \mathrm{Sym}_+(A)$. By ISO2:

$$HN \big/ N \approx H \big/ H \cap N$$

If $H \subseteq N$, then $HN = N$, and $[HN : N] = 1$. Otherwise, there are indirect isometries in H, such as some $f \in H \setminus N$. There are two cosets of N in HN: direct coset N and indirect coset fN - the index of N in HN is 2. ■

1.1.1 First Form of the Crystallographic Restriction Theorem

In this early stage, it is worth to state Barlow elegant proof of the first form:

Theorem 1.1 (First Form of the Crystallographic Restriction Theorem) *Given a lattice L, if $S \in \mathrm{Sym}(L)$ is a rotation, then $\mathrm{o}(S) \in \{1, 2, 3, 4, 6\}$. Or: the minimal angle of a rotation can be $\frac{2\pi}{n}$ and $n \in \{1, 2, 3, 4, 6\}$.*

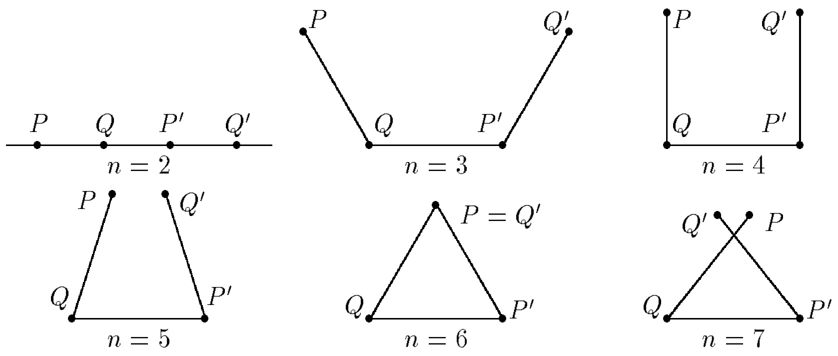


Figure 1.1: First Form of the Crystallographic Restriction Theorem

Proof: (See Figure 1.1) Let P be a center of rotation of period n. The remaining symmetry operations of the lattice transform P into inifitely many other centers of rotation of the same period. Let Q be one of these other centers at the least possible distance from P. A third center, P', is derived from P by rotation through $\frac{2\pi}{n}$ about Q; and a forth, Q', is derived from Q by rotation through $\frac{2\pi}{n}$ about P'. Of course, the segments $\overline{PQ}, \overline{QP'}, \overline{P'Q'}$ are all equal. It may happen that P and Q' coincide; then $n = 6$. In all other cases, since Q was chosen at the least possible distance from P, we must have $\mathrm{d}(P, Q') \geq \mathrm{d}(P, Q)$; therefore $n \leq 4$. (If $n = 4$, $PQP'Q'$ is a square. If $n = 5$, PQ' is obviously shorter than PQ. If $n > 6$, PQ crosses $P'Q'$, but it is no longer necessary to use Q': we already have $\mathrm{d}(P, P') < \mathrm{d}(P, Q)$, which is sufficiently absurd.) ■
An immidiate corollary would be:

Corollary 1.1.3 *Given a lattice L, if $S \in \mathrm{Sym}(L)$ is a rotation with the maximal period, then* $\mathrm{Sym}_+(L) = \langle S \rangle \ltimes T(L)$

Proof: First $T(L)$ is normal, and $\langle S \rangle \ltimes T(L) = \langle S \rangle \cdot T(L)$. If $f \in \langle S \rangle \ltimes T(L)$, then $f = ST$ where $S \in \langle S \rangle, T \in T(L)$. But $S, T \in \mathrm{Sym}_+(L)$, and therefore $f \in \mathrm{Sym}_+(L)$.
Vice-versa, if $f \in \mathrm{Sym}_+(L)$, then f is a rotation or a translation. If $f \in T(L)$, we are done. Otherwise, f is a rotation through α with P as the center. The period of the rotation $S_{O,-\alpha}$ cannot excced $o(S)$, therefore $S_{O,-\alpha} \circ f$ is a translation $T \in T(L)$. Thus, $f = S_{O,\alpha} \circ T$ and its period is the same as of $S_{O,-\alpha}$. ■
It is important to note that Barlow's proof, as state here, used the drawings - you "see" that the argument is true. We will not use this theorem or its corollary, but we will prove this "right" in the next chapter.

1.2 Tessellation Groups

In this section, we will discuss the basic properties of tessellation groups and tiles - these are the mathimatical name for: "how to tile" and "what to tile with".

Definition 1.2.1 (Tessellation Group, Tile, Fundamental Region) *Given a group $G < \mathrm{Isom}(\mathbb{E}^2)$, a connected compact $P \subseteq \mathbb{E}^2$. G shall be called a **tesselatio group**, and P - a **tile** if and only if:*

(GP1)
$$\bigcup_{Q \in G \cdot P} = \mathbb{E}^2$$

(GP2)
$$\forall_{g,h \in G} \quad g(\mathring{P}) \cap h(\mathring{P}) \neq \emptyset \Rightarrow g(P) = h(P)$$

*P is called a **fundamental-region** if and only if:*

$$\forall_{g,h \in H} \quad g(\mathring{P}) \cap h(\mathring{P}) \neq \emptyset \Rightarrow g = h$$

From now-on, we will denote the orbit of P under G by $\mathcal{P}$:

$$\mathcal{P} \triangleq G \cdot P = \{g(P) | g \in G\}$$

and for every $Q \in \mathcal{P}$, we can choose some $g_Q \in G$ such that $g_Q(P) = Q$.
From the definition of tessellation group and tiles, it is clear that two tiles (two copies of P) can be either equal or have distinct interior, and if two tiles' interiors have any point in common, then the tiles are equal. Another trivial argument is that a tile has space: P is a compact with non-empty interior. This brings-up the following lemma:

Lemma 1.2.1 *Given a tessellation group G with a tile P, and any ball B in $\mathbb{E}^2$. The number of tiles in B is finite.*

Proof: Let's define: $H \triangleq \{g_Q | Q \in \mathcal{P}, Q \subseteq B\}$. Given any $x_0 \in \overset{\circ}{P}$, then there exists $r_0 > 0$ such that $B_d(x_0, r_0) \subseteq \overset{\circ}{P}$. Thus the set $A \triangleq H \cdot x_0$ is $2r_0$-discreet in the compact $\overline{B}$. Therefore A is finite, and by GP2: $|A| = |H \cdot P|$, and we are done. ■
neighbor tiles Q and P hold: $P \cap Q$ is not empty and $\overset{\circ}{P} \cap \overset{\circ}{Q}$ is empty. $P \neq Q$ and $(P \setminus \overset{\circ}{P}) \cap (Q \setminus \overset{\circ}{Q}) \neq \emptyset$. The last lemma gives us:

Corollary 1.2.2 *Given a tessellation group G with a tile P. The number of neighbor tiles of P is finite.*

Proof: Given any $a \in \overset{\circ}{P}$. The ball $B_{\mathrm{d}}(a, 3\,\mathrm{diam}(P))$ contains all the neighbors of P. ■
Another property can also be proved with the previous lemma:

Preposition 1.2.3 *Given a tessellation group P with a tile P. Then $\mathrm{Sym}_G(Q)$ is a finite subgroup of G for every $Q \in \mathcal{P}$.*

Proof: It is sufficient to prove that $\mathrm{Sym}_G(P)$ is finite, because $\mathrm{Sym}_G(Q) = {}^{g_Q}\,\mathrm{Sym}_G(Q)$ for every $Q \in \mathcal{P}$. By the Yung Theorem, there exists a point a such that $\mathrm{Sym}_G(P) \subseteq \mathrm{Stab}_G(a)$. Let's take another tile $P' \in \mathcal{P}$ such that the corresponding a' (by the Yung Theorem) is distinct from a. If $\mathrm{Sym}_G(P)$ is applied on $\mathcal{P}$, by Orbit-Stabilizer Theorem we get:

$$\left| \mathrm{Sym}_G(P) \big/ H \right| = |\mathrm{Sym}_G(P) \cdot P'|$$

where $H \triangleq \mathrm{Sym}_G(P) \cap \mathrm{Sym}_G(P')$ - the Stabilizer of P' under the action of $\mathrm{Sym}_G(P)$ on $\mathcal{P}$. If H and $\mathrm{Sym}_G(P) \cdot P'$ are finite, so $\mathrm{Sym}_G(P)$ is finite as well. $\mathrm{Sym}_G(P) \cdot P'$ is finite because its elements are subsets of the ball $B_{\mathrm{d}}(a, \mathrm{d}(a, a') + \mathrm{diam}(P))$, and H is a subset of the finite group $\mathrm{Sym}_G(\{a, a'\})$. Therefore $\mathrm{Sym}_G(P)$ is finite in G. ■
The intuitive idea that the orbit of a point under the action of a tessellation group has no accumulation points, can be proved using the following lemma:

Lemma 1.2.4 *Given a tessellation group G with a tile P. For every $a \in \mathbb{E}^2$ $P \cap G \cdot a$ is finite.*

Proof: If $a \notin P$, then we can find $a' \in P \cap G \cdot a$ and $G \cdot a = G \cdot a'$. Therefore it is sufficient to prove for every $a \in P$. We want to show that there is a finite number of such $g \in G$ such that $g(a) \in P$.
If $a \in \overset{\circ}{P}$, then so $g(a) \in \mathrm{int}\, g(P) = g(\overset{\circ}{P})$, and $\overset{\circ}{P} \cap g(\overset{\circ}{P})$ is not empty. Thus by GP2, $g \in \mathrm{Stab}_G(P)$, which is finite. The problem arises when $a \in P \setminus \overset{\circ}{P}$.
Let $Q \triangleq g(P)$. First, $g(a) \in P \cap g(P)$ and $(g_Q^{-1} \circ g)(P) = P$, which is the same as $g_Q^{-1} \circ g \in Stab_G(P)$, or $g \in g_Q\,\mathrm{Stab}_G(P)$. The number of neighbor tiles Q is finite, so the number of cosets $g_Q\,\mathrm{Stab}_G(P)$, where Q is a neighbor, is finite as well. Therefore g is selected from a finite union of finite sets, and thus there is a finite number of options to select g. ■
And the final property to be considered is:

Preposition 1.2.5 *Given a tessellation group G with a tile P, then there exists $r > 0$ such that $G \cdot a$ is an r-discreet set.*

Proof: Let $B \triangleq B(a, r)$ for some constant $r > 0$. The number of tiles in $B_{\mathrm{d}}(a, r + \mathrm{diam}(P))$ is finite. Therefore $B \cap G \cdot a$ is contained in a finite number of tiles, thus $B \cap G \cdot a$ is finite. ■
Before we continue discussing tessellation groups, we should replace P with something a standard tile, which is easier to research.

1.3 Ford-Region

Temporarily, given any group $G < \text{Isom}(\mathbb{E}^2)$. This group might be or might be not a tessellation group. Anyway, we are trying to define a tile for G:

Definition 1.3.1 (Ford-Region) *Given a group $G < \text{Isom}(\mathbb{E}^2)$. We shall define the* ***Ford-region*** *for $O \in \mathbb{E}^2$ as:*

$$D(G;O) \triangleq \bigcap_{g \in G \setminus \text{Stab}_G(O)} H_{O,g}$$

where:
$$H_{O,g} \triangleq \{P \in \mathbb{E}^2 | \text{d}(P, g(O)) \geq d(P,O)\}$$

$H_{O,g}$ is the closed half-place that contains O and is defined by $\mathcal{H}_{Og(O)}$. [1] *$D(G;O)$ is closed and connected.*

First, let's find-out what are the basic properties of the new creature:

Preposition 1.3.1 *For every $g \in G < \text{Isom}(\mathbb{E}^2)$ and for every $O \in \mathbb{E}^2$, $g(D(G;O)) = D(G;g(O))$.*

Proof: To prove this, we need to prove that $g(H_{O,h}) = H_{g(O),{}^g h}$: the defining lines of the half-planes are equal because:

$$g(\mathcal{H}_{Oh(O)}) = \mathcal{H}_{g(O)gh(O)} = \mathcal{H}_{g(O)({}^g h \circ g)(O)}$$

Now, we can calculate:

$$g(D(G;O)) = g\left(\bigcap_{h \in G \setminus \text{Stab}_G(O)} H_{O,h}\right) = \bigcap_{h \in G \setminus \text{Stab}_G(O)} g(H_{O,h})$$
$$= \bigcap_{h \in G \setminus \text{Stab}_G(O)} H_{g(O),ghg^{-1}}) = \bigcap_{h \in G \setminus \text{Stab}_G(g(O))} H_{g(O),h} \quad \blacksquare$$

Another useful property:

Preposition 1.3.2 *Given $G < \text{Isom}(\mathbb{E}^2)$ and $O \in \mathbb{E}^2$. For every $g \in G \setminus \text{Stab}_G(O)$, the Ford-Regions $D(G;O)$ and $D(G;g(O))$ are neighbors or disjoint.*

Proof: All we have to prove is that if $x \in D(G;O) \cap D(G;g(O))$ then $x \notin \text{int}\, D(G;O) \cup \text{int}\, D(G;g(O))$. By definition of $D(G;O)$, $x \in H_{O,g}$, and by definition of $D(G;g(O)$, $x \in H_{g(O),g^{-1}(O)}$. Thus $x \in \mathcal{H}_{Og(O)}$, and x can be an interior point of neigther $D(G;O)$ nor $D(G;g(O))$. $\blacksquare$

The following property reminds GP1:

Preposition 1.3.3 *Given $G < \text{Isom}(\mathbb{E}^2)$ and $O \in \mathbb{E}^2$. If $G \cdot O$ is closed in $\mathbb{E}^2$, then $\bigcup_{g \in G} D(G;g(O)) = \mathbb{E}^2$.*

Proof: It is trivial that $\bigcup_{g \in G} D(G,g(O)) \subseteq \mathbb{E}^2$.

Given $x \in \mathbb{E}^2$. The distance $\text{d}(x, G \cdot O)$ is $\text{d}(x, g(O))$ for some $g \in G$ because $G \cdot O$ is closed. We want to prove that $x \in D(G;g(O))$: by the choice of g, $\text{d}(x,g(O)) \leq \text{d}(x,h(O))$ for every $h \in G$ - even when $h(O) \neq g(O)$. Thus $x \in D(G;O)$. $\blacksquare$

Now we have the tools to combine tessellation groups with their Ford-regions. Now we will prove that we can replace the tile with some Ford-region (not just any Ford-region ...).

Theorem 1.2 (Topological Definition for Tessellation Groups) *Given a group $G < \text{Isom}(\mathbb{E}^2)$. G is a tessellation group with some tile $P \subseteq \mathbb{E}^2$ if and only if:*

1. *There exists a point O, whose orbit is closed;*

2. $\text{int}\, D(G;O) \neq \emptyset$;

[1] If $g \in \text{Stab}_G(O)$, then $H_{O,g} = \mathbb{E}^2$.

3. $D(G;O)$ is bounded.

Proof: First, we shall prove the opposite direction: if all three requirements hold, then $P = D(G;O)$ is a connected compact, and GP1 holds. Given $g, h \in G$, $\text{int}\, D(G;h(O)) \cap \text{int}\, D(G;g(O)) \neq$, and therefore $g(O) = h(O)$ and $D(G;g(O)) = D(G;h(O))$ - GP2 holds.
If G is a tessellation group with some tile $P \subseteq \mathbb{E}^2$, then P has a center O by the Yung Theorem. The orbit of O is the set of all centers, by the Yung Theorem, of all elements of $\mathcal{P}$. Thus $G \cdot O$ is closed and discreat, and therefore $\mathring{D}(G;O)$ is not empty. To prove the third property, we will bound $D(G;O)$ in a parallelogram:
Given $g \in G \setminus \text{Sym}_G(P)$, then $D(G;O) \subseteq H_{O,g} \cap H_{O,g^{-1}}$. Given another $h \in G \setminus \text{Sym}_G(O)$ such that the lines $h(O)O, g(O)O$ are not parallel (if such h does not exist, then it contradicts GP1). Finally:

$$D(G;O) \subseteq H_{O,g} \cap H_{O,g^{-1}} \cap H_{O,h} \cap H_{O,h^{-1}}$$

$D(G;O)$ is a subset of a bounded parallelogram. ■

1.4 "Re-tessellation" using Ford-regions

The last theorem simplyfies our discussion about tessellation groups. From now-on, G is a tessellation group with the tile $D(G;O)$, such that: $G \cdot O$ is closed, the interior of the new tile is not empty, and the new tile is bounded.

Preposition 1.4.1 *Given a tessellation group $G < \text{Isom}(\mathbb{E}^2)$ with the tile $D(G;O)$. $O \in \text{int}\, D(G;O)$.*

Proof: We have proved that $G \cdot O$ is an r-discreat set for some small enough $r > 0$. Therefore $D(G;O)$ contains $B_{\text{d}}(O, \frac{r}{2})$. ■

Another trivial agrument should be noted:

Lemma 1.4.2 *Given a tessellation group $G < \text{Isom}(\mathbb{E}^2)$ with the tile $D(G;O)$. $\text{Stab}_G(O) < \text{Sym}(D(G;O))$.*

Proof: A simple calculation would prove this:

$$\begin{aligned}\text{Sym}(D(G,O)) &= \{g \in \text{Isom}(\mathbb{E}^2) | g(D(G;O)) = D(G;O)\} \supseteq \{g \in G | g(D(G;O)) = D(G;O)\} \\ &= \{g \in G | D(G;g(O)) = D(G;O)\} = \text{Stab}_G(O),\end{aligned}$$

The last equality holds because for every $g \notin \text{Stab}_G(O)$ the interiors of $D(G;O)$ and $D(g;g(O))$ are disjoint. ■
It would be easier if O is the center of $D(G;O)$ by the Yung Theorem. Such O should hold the property of as big as possible $\text{Stab}_G(O)$.

Definition 1.4.1 (Singular Point) *A **singular point** P of G is a point such that $\text{Stab}_G(P)$ is not trivial: there exists a $\text{id}_{\mathbb{E}^2} \neq g \in G$ such that $P \in \text{Fix}(g)$.*

The singular points are centers of rotation and the points that lie on mirrors. The basic property of a center of rotation in G is:

Preposition 1.4.3 *Given a tessellation group $G < \text{Isom}(\mathbb{E}^2)$ with the tile $D(G;O)$. A singular point $P \neq O$ of G_+ is not in the interior of $D(G;O)$.*

Proof: Let P be such point. Then there is a rotation $S \in G_+$ such that P is its center. Let's assume that $P \in \text{int}\, D(G;O)$, then $\mathring{D}(G;O)$ and $\mathring{D}(G;S(O))$ are not disjoint: they both contain an open ball with positive radios and center P. $P = O$ if and only if $D(G;S(O)) = D(G;O)$, therefore if P is a singular point of G_+ in $D(G;O)$, then eighter $P = O$ or P is not in $\text{int}\, D(G;O)$. ■
Before continuing exploring the properties of $D(G;O)$ as a tile, let's consider the following groups:

Definition 1.4.2 (C_n, D_n) *Given $n \in \mathbb{N}$, $O \in \mathbb{E}^2$ and a line l such that $O \in l$. Then $C_n \triangleq \left\langle S_{O,\frac{2\pi}{n}} \right\rangle$ and $D_n \triangleq \left\langle S_{O,\frac{2\pi}{n}}, R_l \right\rangle$.*

It is important to note that the choice of O and of l has no meaning at all: C_n and C'_n are always isomorphic, with the mapping: $f \mapsto {}^{T_{O,O'}}f$. D_n and D'_n are also isomorphic, with the mapping: $f \mapsto {}^{g}f$, where g is direct and holds: $g(l) = l'$.
Finally, we can prove the following lemma regarding the center of $D(G;O)$:

Lemma 1.4.4 *Given a tessellation group $G < \text{Isom}(\mathbb{E}^2)$ with the tile $D(G;O)$. If O is a singular point of G_+, then the center of $D(G;O)$ by the Yung Theorem is O.*

Proof: Let $S \in \text{Stab}_G(O) < \text{Sym}(D(G;O))$ be a non-trivial rotation with center O. By Yung Theorem, there is a point O' such that $\text{Sym}(D(G;O)) < \text{Stab}(O')$. Thus $S \in \text{Stab}(O')$. The only possible case is when $O' = O$. ■
Now we can go even further:

Theorem 1.3 *Given a tessellation group $G < \text{Isom}(\mathbb{E}^2)$ with the tile $D(G;O)$. The number of singular points in $D(G;O)$ is finite. More-ever,* $\text{Sym}(D(G;O))$ *is finite and isomorphic to D_n or to C_n.*

Proof: Let S be the set of all singular points of G_+ in $\mathbb{E}^2$. First, it is closed and discreat: the completement is open because if $P \notin S$, then $\mathring{D}(G;P) \cap S = \emptyset$. And, For every $Q \in S$, $\mathring{D}(G;Q) \cap S = \{Q\}$. The set of all singular points in $D(G;O)$: $S_O = S \cap D(G;O)$ is a closed and discreat subset of the compact $D(G;O)$. Thus, S_O is finite.
Now, for the "more-ever" part: Let Γ be the group of all permutations of the set S_O, and let ϖ be the mapping:

$$\begin{cases} \varphi : \text{Sym}_+(D(G;O)) & \rightarrow & \Gamma \\ g & \mapsto & g\big|_{S_0} \end{cases}$$

Let's note that if $P \in S_O$, then $g(P) \in S_O$ for every $g \in \text{Sym}_+(D(G;O))$, because if $S_P \in G$, so ${}^{g}S_P = S_{g(P)} \in G$. Therefore φ is a homomorphism. Now, we will show that φ is $1-1$: it is sufficiant to show that if $\varphi(g) = 1|_{S_O}$, then $g = \text{id}_{\mathbb{E}^2}$. If $P \in S_O \setminus \{O\}$, then the only rotation in $\text{Sym}(D(G;O))$ that leaves P in its place is $\text{id}_{\mathbb{E}^2}$, therefore φ is $1-1$, and $\text{Sym}_+(D(G;O)$ is finite.
If $\text{Sym}_+(D(G;O))$ is cyclic, then $\text{Sym}_+(D(G;O))$ is isomorphic to C_n. Let $f \in \text{Sym}_+(D(G;O))$ be a non-trivial rotation with minimal positive angle α (such f exists because $\text{Sym}_+(D(G;O))$ is finite). Let $g \in \text{Sym}_+(D(G;O))$ be another non-trivial rotation with positive angle β ($\beta \geq \alpha$). There exists a maximal $q \in \mathbb{N}$ such that $q\alpha \leq \beta$. Let $r \triangleq \beta - q\alpha$. The angle of the rotation $f^{-q} \circ g$ is r which is less than α. r can be only 0, and $g = f^q$. Therefore $\langle f \rangle = \text{Sym}_+(D(G;O))$ and $\text{Sym}_+(D(G;O))$ is isomorphic to C_n where $n = \text{o}(f)$.
In light of lemma 1.1.2, $\text{Sym}(D(G;O))$ is finite and isomorphic to C_n or D_n. ■

1.5 The Structure of Lattice-Groups

A lattice-group is the symmetry group of a lattice. This group is a simple example for tessellation groups. Our temporary postulate is that any tessellation group is a lattice-group. Currently, we have no meaning to check wheter this postulate is true or false. Our research of tessellation groups becomes much easier in this case, because it is better defined.
Let L be a lattice in $\mathbb{E}^2$, let $O \in L$ be a constant point, and let $G = \text{Sym}(L)$. We want to use $D(G;O)$ to research G. It is trivial that $D(G;O)$ is bounded:

$$D(G;O) \subseteq H_{O,T_1} \cap H_{O,T_1^{-1}} \cap H_{O,T_2} \cap H_{O,T_2^{-1}}.$$

Another trivial property of $D(G;O)$ is that O is the center of $D(G;O)$ by the Yung Theorem, because $H_O \in \text{Stab}_G(O)$. The first properties to consider are:

Definition 1.5.1 (Translations Sub-group) *Given $G < \text{Isom}(\mathbb{E}^2)$, then $T(G)$ is the group of all translations in G.*

Lemma 1.5.1 *Let L be a lattice, and $G = \text{Sym}(L)$. Then:*

1. $T(L) \cap \mathrm{Sym}(D(G;O)) = \{\mathrm{id}_{\mathbb{E}^2}\}$,

2. $T(G) = T(L)$.

3. $G \subseteq T(L) \cdot \mathrm{Sym}(D(G;O)) = T(G) \cdot \mathrm{Sym}(D(G;O))$,

4. $T(L) = T\big(T(L) \cdot \mathrm{Sym}(D(G;O))\big)$, $\;:\; T(G) = T\big(T(G) \cdot \mathrm{Sym}(D(G;O))\big)$.

Proof: The first statement is trivial: $D(G;O)$ is bounded, and there are no translations in $\mathrm{Sym}(D(G;O))$.
The second statement is easy as well: First, any translation in $T(L)$ is a symmetry of L. Secondly, if $f \in T(G)$, then $T(O) \in L$ and $T = T_{O,T(O)} \in T(L)$.
The third statement is trivial: if $f \in \mathrm{Stab}_G(O)$, then $f \in \mathrm{Sym}(D(G;O))$ (this was proved earlier in lemma 1.4.2). If $f \in G \setminus \mathrm{Stab}_G(O)$, then $f = T_{O,f(O)} \cdot g$ where $g \in \mathrm{Stab}_G(O)$.
To prove the forth statement, it is sufficiat to show that a translation $N \triangleq T(G) \cdot \mathrm{Sym}(D(G;O))$ is a translation in G. Every $f \in N$ can be represented as $T_f \cdot S_f$ where $T_f \in T(G)$ and $S_f \in \mathrm{Sym}(D(G;O))$. If f is a translation, then the translation T_f^{-1} is a symmetry of $D(G;O)$. Therefore it is $\mathrm{id}_{\mathbb{E}^2}$, by the first statement. Thus $T_f = f$. ■
Now, we can prove the following:

Preposition 1.5.2 $\mathrm{Stab}_G(O) = \mathrm{Sym}(D(G;O))$

Proof: It is sufficiant to prove that every $f \in \mathrm{Sym}(D(G;O))$ is in G. If we prove this, then lemma ?? assures us that $f(O) = O$. Let $P \in L \setminus \{O\}$ be a point, we have to prove that $f(P) \in L$.
Let $T_P \triangleq T_{O,P} \in T(G)$. There exists a point $Q \in L$ such that $f(P) \in D(G;Q)$, and let $T_Q \triangleq T_{O,Q}$. The point $P' \triangleq (T_Q^{-1} \circ f)(P)$ is in $D(G;O)$. If $P' = O$, then $f(P) = Q$ and we are done. $T(G) = T(L)$ is a normal subgroup of $T(L) \cdot \mathrm{Sym}(D(G;O))$ by the previous lemma. Thus there exists a translation T_f such that $f \cdot T_P = T_f \cdot f$. Therefore:

$$P' = (T_Q^{-1} \cdot f \cdot T_P)(O) = (T_Q^{-1} \cdot T_P \cdot f)(O) = (T_Q^{-1} \cdot T_f)(O).$$

The translations are in G, therefore $P' \in L \cap D(G;O)$. By $D(G;O)$ contains only one lattice point - O. We get $Q = f(P)$. ■
Now, $T(L) \lhd G$ and $\mathrm{Sym}(D(G;O)) < G$, and we have proved:

Theorem 1.4 (The Structure of Lattice-Groups) *Given a lattice L, $G = \mathrm{Sym}(L)$ and $O \in L$. Then $G = \mathrm{Sym}(D(G;O)) \ltimes T(L)$.* ■

The first symmetry we have discovered in a lattice was H_O. This means that every point in the lattice a singular point in G_+. This argument combined with the last theorem and theorem ??, we get:

Corollary 1.5.3 (The definition of Lattice-Groups) *Given a lattice L and $G = \mathrm{Sym}(L)$. Then $G = C_n \ltimes T(L)$ or $G = D_n \ltimes T(L)$.*

If we apply Barlow's proof, G is isomorphic to one of the following: $C_2 \ltimes \mathbb{Z}^2$, $D_2 \ltimes \mathbb{Z}^2$, $D_4 \ltimes \mathbb{Z}^2$ or $D_6 \ltimes \mathbb{Z}^2$.

Chapter 2

The Crystallographic Restriction Theorem

This chapter is dedicated to the proof of the following Theorem:

Theorem 2.1 (The Crystallographic Restriction Theorem)
There exists exactly 17 isomorphism equivalence-classes of tessellation groups in the Euclidean Plane. They are:

Symbol	*Generators*	*tile*
p1	*Two translations*	*parallelogram*
p2	*Three half-turns at the vertices of a rectangle*	*rectangle*
p3	*Two rotations through* $\frac{2\pi}{3}$	*rhomb with* $\frac{2\pi}{6}, \frac{2\pi}{3}$ *angles*
p4	*Half-turn and a rotation through* $\frac{2\pi}{4}$	*square*
p6	*Half-turn and a rotation through* $\frac{2\pi}{3}$	*a* $(\frac{2\pi}{3}, \frac{2\pi}{12}, \frac{2\pi}{12})$ *triangle*
pm	*Two parallel reflections and a translation whose direction is parallel to the mirrors*	*rectangle*
pg	*Two parallel glides*	*rectangle*
cm	*Parallel Glide and reflection*	
pmm	4 *reflections in the* 4 *sides of a rectangle*	*rectangle*
pmg	*A reflection and* 2 *half-turns*	*rectangle*
pgg	2 *perpendicular glide reflections*	*square*
cmm	2 *perpendicular reflections and a half-turn*	
p4m	*Reflections in the three sides of a* $(\frac{2\pi}{4}, \frac{2\pi}{8}, \frac{2\pi}{8})$ *triangle*	*a* $(\frac{2\pi}{4}, \frac{2\pi}{8}, \frac{2\pi}{8})$ *triangle*
p4g	*A reflection and a rotation through* $\frac{2\pi}{4}$	*a* $(\frac{2\pi}{4}, \frac{2\pi}{8}, \frac{2\pi}{8})$ *triangle*
p3m1	*A reflection and a rotation through* $\frac{2\pi}{3}$	*a* $(\frac{2\pi}{3}, \frac{2\pi}{12}, \frac{2\pi}{12})$ *triangle*
p31m	3 *reflections in the three sides of an equilateral triangle*	*an equilateral triangle*
p6m	*Reflections in the three sides of a* $(\frac{2\pi}{4}, \frac{2\pi}{6}, \frac{2\pi}{12})$ *triangle*	*a* $(\frac{2\pi}{4}, \frac{2\pi}{6}, \frac{2\pi}{12})$ *triangle*

The order of the symbols in the table represent the order of the proof.

2.1 The Connection to Lattice-Groups

Before proving the Theorem, we shall consider representing tessellation groups as subgroups of lattice-groups

Theorem 2.2 *Given a tessellation group* $G < \mathrm{Isom}(\mathbb{E}^2)$ *with a tile* P*, and let* O *be the center of* P *(by Yung's Theorem). Then* $L \triangleq T(G) \cdot O$ *is a lattice and* $G_+ < \mathrm{Sym}_+(L)$*.*

Proof: The first step is to replace P with $D(G; O)$, where O is the center of P (by Yung's Theorem). Now, $G \cdot O$ is closed and discreat, and we have to prove that $T(G)$ is constructed by two translations whose directions are not parallel.

If $T(G)$ is trivial, then G_+ contains only rotations with a common center. The mirrors of the reflections in G meet in the center of all rotations. Therefore all elements of $G \cdot D(G; O)$ is inside a ball - contradiction to GP1. Thus $T(G)$ is not trivial and contains at least one translation.
Let assume that the directions of all translations in $T(G)$ are parallel to a line l. Let $t \in T(G)$ and $g \in G \setminus T(G)$, then ${}^g t \in T(G)$. Therefore the mirrors of all reflections are perpendicular to l, and the axises of all glides are equal and parallel to l, and thus l can be replaced with the axis of all glides. If $f \in G_+$ is a rotation and $t \in T(G)$, then ${}^f t \in T(G)$ and $f = \mathrm{id}_{\mathbb{E}^2}$ or $f = H_Q$ where $Q \in l$. Therefore $G \cdot D(G; O)$ lies on a constant-width strip around l - contradicting to GP1.
Till now, we have proved that there are at least two non-parallel translations in $T(G)$. Since $G \cdot O$ is closed and discreat, there exists $T_1 \in T(G) \setminus \{\mathrm{id}_{\mathbb{E}^2}\}$ with minimal $|T_1|$, and there exists $T_2 \in T(G) \setminus \langle T_1 \rangle$ with minimal $|T_2|$. We want to prove that $T(G) = \langle T_1, T_2 \rangle$.
It is trivial that $\langle T_1, T_2 \rangle \subseteq T(G)$. To prove the other direction, let $L' \triangleq T(G) \cdot O$ and $L \triangleq \langle T_1, T_2 \rangle \cdot O$, and we want to show that $L = L'$. Let's assume that there exists $y \in L' \setminus L$. Since L is losed and discreat, there exists $x \in L$ which is the closest one to y. Let $T \triangleq T_{x,y}$. If $|T| \geq \min(|T_1|, |T_2|)$, then there is closer point in L to y. Otherwise, T is smaller than T_1 and T_2 - which contradicts the minimality of T_1, T_2. We have a lattice L and $T(L) = T(G)$.
The last argument to be proved is that $G_+ < \mathrm{Sym}_+(L)$. It is sufficiant to show that $\mathrm{Stab}_{G_+}(O) < \mathrm{Sym}_+(L)$. Let $P \in L$, $f \in \mathrm{Stab}_{G_+}(O)$, and $t \triangleq T_{O,P} \in T(G)$. To prove that $f(P) \in L$, we will prove that $T_{P,f(P)} \in T(G)$. But $T_{P,f(P)} = {}^f t \circ t^{-1}$, and:

$$({}^f t \circ t^{-1})(P) = ({}^f t \circ (t^{-1} \circ t))(O) = (f \circ t \circ f^{-1})(O) = (f \circ t)(O) = f(P)$$

Secondly, $T(G) < \mathrm{Sym}(L)$. If $g \in G_+$ then there exists $f \in \mathrm{Stab}_{G_+}(O)$ and $t \in T(G)$ such that $g = t \circ f$, and therefore $g \in \mathrm{Sym}_+(L)$. ■
The last argument proves the corollary -

Corollary 2.1.1 *Given a tessellation group $G < \mathrm{Isom}(\mathbb{E}^2)$ with a tile P, then $G_+ = \langle S \rangle \ltimes T(G)$ where $S \in G_+$ is the minimal rotation whose center is the tile's center (by Yung's Theorem).*

Now, we are able to prove the Theorem. The theorem is divided into parts: the first part deals with direct tessellation groups and the second part deals with the other tessellation groups.

2.2 Direct Tessellation Grups: $G = G_+$

We are going to prove the first form the Crystallographic Restriction Theorem, but here we are handling a direct tessellation group G, which is a subgroup $\mathrm{Sym}(L)$.
Proof: Given a tessellation group $G < \mathrm{Isom}(\mathbb{E}^2)$ that holds: $G = \langle S_O \rangle \ltimes T(G) < \mathrm{Sym}(L)$ where S_O is the minimal rotation and $O \in L$. Let $G' \triangleq G \setminus T(G)$. If $G' = \emptyset$, then there are no rotations and we have got the equivalence class **p1** whose tessellation groups are isomorphic to $T(L)$ and to $\mathbb{Z} \oplus \mathbb{Z}$. If the maximal order of elements in G' is 2, then the rotations of G' are half-turns, and we have got **p2**.
Let $S_A \in G'$ be a rotation whose center is $A \in L$ and whose order is bigger than 2. Let $B \in L$ the closest point to A, and according to the choice of T_1, T_2 from theorem 2.2, $B = T_1(A)$. Let $S_B \in G'$ be a rotation whose center is B and whose order is bigger than 2. Let $S_C \triangleq (S_A \circ S_B)^{-1} \in G'$. The angle of S_C is $2\pi \left(1 - \frac{1}{\mathrm{o}(S_A)} - \frac{1}{\mathrm{o}(S_B)}\right)$, but it is also $\frac{2\pi}{\mathrm{o}(S_C)}$. Therefore:

$$\frac{1}{\mathrm{o}(S_A)} + \frac{1}{\mathrm{o}(S_B)} + \frac{1}{\mathrm{o}(S_C)} = 1$$

Since an order is natural, there are only three sollutions to the equation:

$\mathrm{o}(S_A) = \mathrm{o}(S_B) = \mathrm{o}(S_C) = 3$**:** We have got **p3**.

$\mathrm{o}(S_A) = \mathrm{o}(S_B) = 4, \mathrm{o}(S_C) = 2$**:** We have got **p4**.

$\mathrm{o}(S_A) = 3, \mathrm{o}(S_B) = 6, \mathrm{o}(S_C) = 2$**:** We have got **p6**. ■

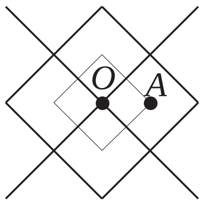


Figure 2.1: Proof of Lemma 2.3.1: $A \notin T'(G) \cdot O$

2.3 Indirect Tessellation Groups ($G \neq G_+$)

Indirect tessellation groups can be divided into five groups, according to the structure of G_+. From now, we will denote $A \triangleq T_1(O)$ and $B \triangleq T_2(O)$. In order to make the proof easier, we shall prove:

Lemma 2.3.1 *Given a tesslation group $G < \text{Isom}(\mathbb{E}^2)$ such that $G \neq G_+$. If $f \in G \setminus G_+$, then one or the other, but no both, of the following conditions holds:*

A1: ${}^fT_1 = T_1$ *and* ${}^fT_2 = T_2^{-1}$;

A2: ${}^fT_1 = T_2$ *and* ${}^fT_2 = T_1$.

Proof: First, we show that A1 and A2 are incompatible: we assume that $T(G) = \langle T_1, T_2 \rangle = \langle T_1', T_2' \rangle$ and:

$$ {}^fT_1 = T_2 \quad \wedge \quad {}^fT_2 = T_1 \quad \wedge \quad {}^fT_1' = T_1' \quad \wedge \quad {}^fT_2' = T_2'^{-1} $$

If $T_1' = T_1^h T_2^k$, then -

$$ T_1' = {}^fT_1' = {}^fT_1^h {}^fT_2^k = T_1^k T_2^h $$

Therefore $h = k$ and $T_1' = (T_1T_2)^h$. If $T_2' = T_1^m T_2^n$, then -

$$ T_2'^{-1} = {}^fT_2' = {}^fT_1^m {}^fT_2^n = T_1^n T_2^m $$

Thus $m = -n$, and $T_2' = (T_1T_2^{-1})^m$. We have shown that $T_1', T_2' \in T'(G) \triangleq \langle T_1T_2, T_1T_2^{-1} \rangle$. But $T'(G)$ is a proper subgroup of $T(G)$, because the lattice $T'(G) \cdot O$ does not contain A - A is on the border of $D(T'(G), O)$.

We now suppose T_1, T_2 chosen as before: $|T_1|$ is minimal in $T(G) \setminus \{\text{id}_{\mathbb{E}^2}\}$ and $|T_2|$ is minimal in $T(G) \setminus \langle T_1 \rangle$. After replacing T_1 by T_1^{-1} if necessary, we may assume that $\theta \triangleq \angle(A, O, B) \leq \frac{2\pi}{4}$. The minimality of $|T_2|$ now requires $\text{d}(O, B) \leq \text{d}(A, B)$, and thus B is in the half-plane that contains O and is defined by $\mathcal{H}_{OA}$. Therefore -

$$ \frac{2\pi}{6} \leq \theta \leq \frac{2\pi}{4} $$

Let $l \triangleq OA$. We shall define $h(g) \triangleq \text{d}(O, l(g(O)))$, and $v(g) \triangleq \text{d}(g(O), l)$. $h(g)$ is the distance between O and the image of $g(O)$ on l, and $v(g)$ is the distance between $g(O)$ and l. It is trivial that $v(T_1 \circ g) = v(g)$ because a translation parallel to l does not change the distance between $g(O)$ and l. Another property of $v(g)$ is that $v(T_2) = v(T_2^{-1})$.

Let suppose that ${}^fT_1 = T_1$. This requires that the axis (or mirror) of f is parallel to l. If $\theta = \frac{2\pi}{4}$, then ${}^fT_2 = T_2^{-1}$ and A1 holds. If $\theta < \frac{2\pi}{4}$, then $v({}^fT_2) = v(T_2^{-1} \circ T_1)$, and $h({}^fT_2) = h(T_2)$. Since $0 < h(T_2) \leq \frac{1}{2}|T_1|$, $\frac{1}{2}|T_1| \leq h(T_2^{-1} \circ T_1) < |T_1|$. We conclude that ${}^fT_2 = T_2^{-1} \circ T_1$. Let $T_1' \triangleq T_2$ and $T_2' \triangleq T_2^{-1} \circ T_1$, then $\langle T_1', T_2' \rangle = T(G)$, f exchanges T_1' and T_2' and A2 holds.

Suppose next that ${}^fT_1 = T_1^{-1}$, whence the axis of f is perpendicular to l. If $\theta = \frac{2\pi}{4}$, then ${}^fT_2 = T_2$ and A1 holds. If $\theta < \frac{2\pi}{4}$, then $\overline{({}^fT_2)(O)B}$ is parallel to l and its length $d = 2h(T_2) \leq |T_1|$. The minimality of $|T_1|$ requires $d = |T_1|$, and therefore ${}^fT_2 = T_1^{-1} \circ T_2$. Now f exchanges T_2 and $T_1^{-1} \circ T_2$, which can be the generators of $T(G) = \langle T_1^{-1} \circ T_2, T_2 \rangle$ and A2 holds.

Suppose finally that ${}^fT_1 \notin \{T_1, T_1^{-1}\}$, whence ${}^fT_1$, of length $|{}^fT_1| = |T_1|$, is not a power of T_1. We take $T_2 = {}^fT_1$, and A2 holds. ■

Now we are ready to find the other 12 isomorphism-classes. The proof is divided into 5 cases: for each possible direct tessellation subgroup.

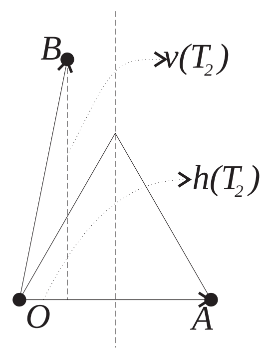


Figure 2.2: Proof of Lemma 2.3.1: $h(g)$ and $v(g)$

${}^fT_1 = T_1$ ${}^fT_1 = T_1^{-1}$

Figure 2.3: Proof of Lemma 2.3.1: 2 possible cases

2.3.1 Case 1: There are no rotations in G_+: $G \neq G_+ = T(G)$

Proof: The first case is $G \neq G_+ = T(G)$. If $f_1, f_2 \in G \setminus G_+$, then $f_2 = T \circ f_1$ for some $T \in G_+$. Therefore ${}^{f_1}T = {}^{f_2}T$ for every $T \in G_+$. In short, all choices of f yield the same automorphism of G_+.

If G contains any reflection, we choose f to be a reflection. If A1 holds, then the mirror of f is parallel to the direction of T_1, and we get **pm**. The generators are the translation T_1, and the reflections $f, f^{-1} \circ T_2$.

If A2 holds, then $|T_1| = |T_2|$ and f exchanges them. The lattice of such tessellation group must be square, while $O \in l$ and $\angle(OA, l) = \frac{2\pi}{8}$, and we get **cm**. The generators are the reflection f and a glide g such that $g^2 = T_1 \circ T_2$ and the axis of g and mirror of f are distinct and parallel.

Now, suppose that G does not contain reflections, whence G contains translations and glides. Let $f \in G$ be a glide, thus $f^2 \in T(G) \setminus \{\mathrm{id}_{\mathbb{E}^2}\}$. If A1 holds, and $f^2 = T_1$, then the axis of f is parallel to the direction of T_1 and its length is $\frac{1}{2}|T_1|$. In order to prove that this tessellation group is new, we need to prove that G does not contain reflections. Since G_+ is a normal subgroup, every $g \in G \setminus G_+$ can be represented as $g = f \circ T_1^m \circ T_2^n$, and g^2 is:

$$\begin{aligned} g^2 &= (f \circ T_1^m \circ T_2^n)^2 = f \circ T_1^m \circ T_2^n \circ f \circ T_1^m \circ T_2^n = f \circ T_1^m \circ f T_2^{-n} \circ T_1^m \circ T_2^n \\ &= f^2 \circ T_1^{2m} = T_1^{2m+1} \neq \mathrm{id}_{\mathbb{E}^2} \end{aligned}$$

g is not a reflection, and we got **pg**, whose generators the glides g_1, g_2 where $g_1^2 = T_1$ and $g_2 = g_1 \circ T_2$.

Finally, if G does not contain any reflections and A2 holds, let $f \in G \setminus G_+$ be a glide such that $f^2 = T_1 \circ T_2$, and let $g = T_1^{-1} \circ f$. g^2 is:

$$\begin{aligned} g^2 &= (T_1^{-1} \cdot f)^2 = T_1^{-1} \circ f \circ T_1^{-1} \circ f = T_1^{-1} \circ {}^fT_2^{-1} \circ f^2 \\ &= (T_1 \circ T_2)^{-1} \circ (T_1 \circ T_2) = \mathrm{id}_{\mathbb{E}^2} \end{aligned}$$

g is a reflection, constradicting the assumption that g does not contain any reflections. ■

Figure 2.4: The basic symmetries of $G \neq G_+ = T(G)$, f is a reflection

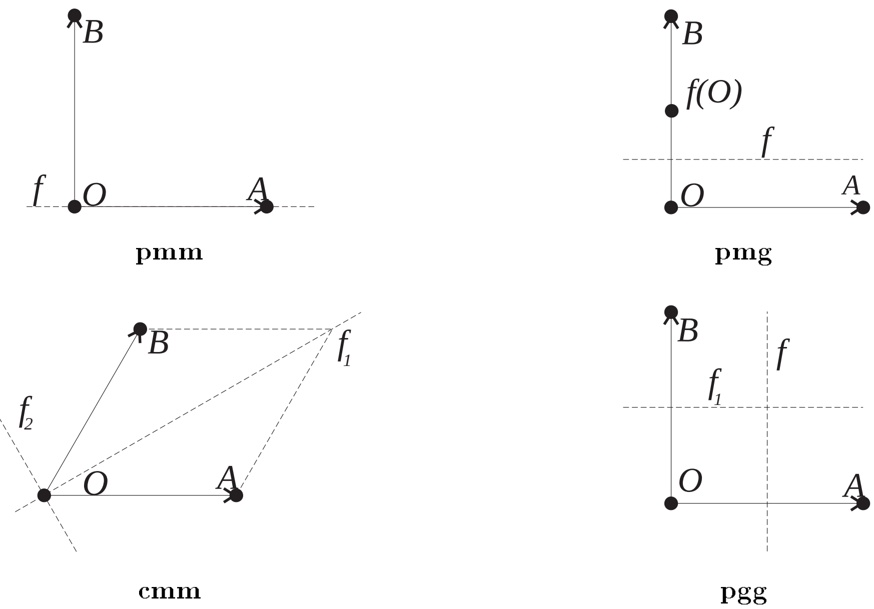


Figure 2.5: Basic symmetries in $G \neq G_+ = \langle H_O \rangle \ltimes T(G)$

2.3.2 Case 2: There are Half-Turns in G_+: $G \neq G_+ = \langle H_O \rangle \ltimes T(G)$

Proof: Let $S = H_O$. It is trivial that ${}^S T = T^{-1}$ for every $T \in T(G)$.

Suppose G contains a reflection f and A1 holds, whence ${}^f T_1 = T_1$ and ${}^f T_2 = T_2^{-1}$. This is possible only if the directions of T_1, T_2 are perpendicular, and the mirror of f of is parallel to the direction of T_1. Let $f_1 \triangleq f \circ S$, and we get an isometry whose axis is parallel to the direction of T_2, and ${}^{f_1} T_2 = T_2$. We may assume that $f_1^2 \in \{\mathrm{id}_{\mathbb{E}^2}, T_2\}$.

Both cases can be made possible in a square lattice. For the first, let the mirror of f be OA, and f_1 is reflection with mirror OB, and we have got **pmm**. For the second, let the mirror of f, l', be parallel to OA and $f(O)$ is the midpoint $\overline{OB}$. In this case f_1 is a glide, $f_1^2 = T_2$, and we have got **pmg**.

Suppose G contains a reflection f and $A2$ holds, whence the mirror of f is parallel to the direction of $T_1 \circ T_2$. Let $f_1 \triangleq f \circ S$, thus:

$$\begin{aligned} {}^{f_1} T_1 &= (f \circ S) \circ T_1 \circ (S \circ f) = \left({}^f T_1\right)^{-1} = T_2^{-1} \\ {}^{f_1} T_2 &= (f \circ S) \circ T_2 \circ (S \circ f) = \left({}^f T_2\right)^{-1} = T_1^{-1} \\ {}^{f_1}\left(T_1^{-1} \circ T_2\right) &= \left({}^{f_1} T_1\right)^{-1} \circ {}^{f_1} T_2 = T_1^{-1} \circ T_2 \end{aligned}$$

The axis of f_1 is parallel to the direction of $T_1^{-1} \circ T_2$. We may assume that $f_1^2 \in \{\mathrm{id}_{\mathbb{E}^2}, T_1^{-1} \circ T_2\}$. If $f_1^2 = T_1^{-1} \circ T_2$, and $f_2 \triangleq T_1 \circ f$, then:

$$f_2^2 = T_1 \circ (f_1 \circ T_1 \circ f_1) = (T_1 \circ T_2^{-1}) \circ f_1^2 = (T_1 \circ T_2^{-1}) \circ (T_1^{-1} \circ T_2) = \mathrm{id}_{\mathbb{E}^2}$$

f_2 is a reflection. If f_1 is a reflection, then f_2 becomes $f \circ S$. In both cases, the axises are perpendicular, while the axis of f_1 is parallel to the direction of $T_1 \circ T_2$ and the axis of f_2 is parallel to the direction of $T_1^{-1} \circ T_2$. Therefore, we have got **cmm**.
Now, suppose G does not have any reflections. If A1 holds, we may assume that $f^2 = T_1$. If $f_1 \triangleq f \circ S$, then:

$$^{f_1}T_1 = {}^f\left({}^S T_1\right) = {}^f\left(T_1\right)^{-1} = T_1^{-1}$$
$$^{f_1}T_2 = {}^f\left({}^S T_2\right) = {}^f\left(T_2\right)^{-1} = T_2$$

We may likewise suppose that $f_1^2 = T_2$. To check that this G does not contain any reflections, we will check: $(T_1^m \circ T_2^n \circ f)^2$:

$$(T_1^m \circ T_2^n \circ f)^2 = T_1^{2m+1} \neq \mathrm{id}_{\mathbb{E}^2}$$

And for $(T_1^m \circ T_2^n \circ f_1)^2$:

$$(T_1^m \circ T_2^n \circ f_1)^2 = T_2^{2n+1} \neq \mathrm{id}_{\mathbb{E}^2}$$

The lattice of this tessellation group is a square one, while the axises of the glides f, f_1 are parallel to T_1, T_2 accordingly, and the length of the glides is $\frac{1}{2}|T_1| = \frac{1}{2}|T_2|$. We have got **pgg**.
If A2 holds, we may assume that the glide f holds $f^2 = T_1 \circ T_2$. $f_1 = f \circ S$ is a glide with $f_1^2 = T_1^{-1} \circ T_2$. But $f_2 = T_1 \circ f_1$ is a reflection, contradicting to hypothesis. ■

2.3.3 Case 4: There are Rotations through $\frac{2\pi}{4}$ in G_+: $G \neq G_+ = \left\langle S_{O,\frac{2\pi}{4}} \right\rangle \ltimes T(G)$

Proof: Let $S \triangleq S_{O,\frac{2\pi}{4}}$, ${}^S T_1 = T_2$ and ${}^S T_2 = T_1^{-1}$, whence the lattice is square. If $f \in G \setminus G_+$ satisfies A1 and $f_1 \triangleq f \circ S$, then:

$$^{f_1}T_1 = {}^f\left({}^S T_1\right) = {}^f T_2 = T_1^{-1}.$$
$$^{f_1}T_2 = {}^f\left({}^S T_2\right) = {}^f T_1^{-1} = T_2^{-1}$$

Hence f_1 satisfies A2 for $T(G) = \langle T_1, T_2^{-1} \rangle$. Therefore A1 and A2 coincide, and we may assume that f satisfies A1, and f is a reflection or a glide.
The set $\{T_1, T_1^{-1}, T_2, T_2^{-1}\}$ is uniquely determined by the facts that S permutes them cyclically and that they generate T:

$$S(T_1, T_2, T_1^{-1}, T_2^{-1}) = (T_2, T_1^{-1}, T_2^{-1}, T_1)$$

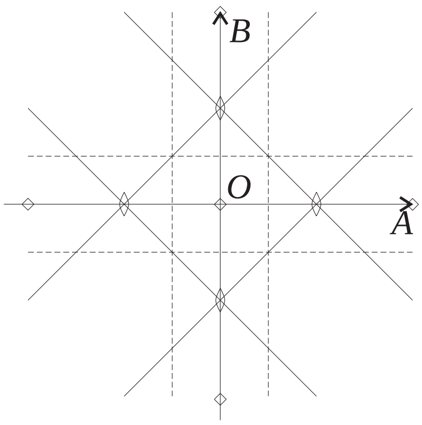
We show that if G contains a reflection parallel to one of T_1, T_2 then it also contains parallel to the other. By symmetry it suffices to consider the case that G contains a reflection f parallel to the direction of T_1, hence satisfying A1. If the axis of f_1 is parallel of the direction of $T_1^{-1} \circ T_2$, then $f_1^2 = (T_1^{-1} \circ T_2)^h$ for some $h \in \mathbb{Z}$. The axis of $T_2^{-h} \circ f_1$ is parallel to the axis of f_1, and to the direction of $T_1^{-1} \circ T_2$. $T_2^{-h} \circ f_1$ is a reflection, because -

$$(T_2^{-h} \circ f_1)^2 = (T_1^{-h} \circ T_2^h) \circ f_1^2 = (T_1^{-1} \circ T_2^1)^{h-h} = \mathrm{id}_{\mathbb{E}^2}$$

We have got 2 reflections whose mirrors meet at O, the angle between the mirrors is $\frac{2\pi}{8}$, and their product is S. Thus we have got **p4g**. Figure 2.6 shows all symmetries of **p4g**. Suppose G contains a glide f whose axis is parallel to the direction of T_1, and does not contain reflections whose mirrors are paralllel to the axis of f. In this case, $f \circ S$ is a reflection or ${}^f S = S^{-1}$. This group is in **p4m**.

2.3.4 Case 3: There are Rotations through $\frac{2\pi}{3}$ in G_+: $G \neq G_+ = \left\langle S_{O,\frac{2\pi}{3}} \right\rangle \ltimes T(G)$

Proof: Now $S \triangleq S_{O,\frac{2\pi}{3}}$, ${}^S T_1 = T_1^{-1} \circ T_2$ and ${}^S T_2 = T_1^{-1}$.



$\diamond$ represents the center of a rotation through $\frac{2\pi}{4}$, $\lozenge$ represents the center of a half-turn, solid lines are mirrors of reflections, and broken lines are axes of glide reflections.

Figure 2.6: Basic symmetries in **p4g**

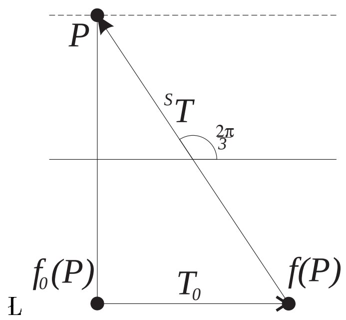


Figure 2.7: The argument for $G \neq G_+ = \left\langle S_{O,\frac{2\pi}{3}} \right\rangle T(G))$

(Look at figure 2.7) First, we show that if G contains a glide with axis l then there is a reflection f_1 with mirror l_1 parallel to l. Let $f \triangleq f_0 \circ T_0 \in G$ be a glide, with a reflection $f_0 \in \mathrm{Isom}(\mathbb{E}^2)$ and translation $T_0 \in \mathrm{Isom}(\mathbb{E}^2)$ whose direction is parallel to the axis of f. Therefore $T \triangleq f^2 = T_0^2 \in T(G)$. Let l_1 be a line parallel to l such that $\mathrm{d}(l, l_1) = \frac{\sqrt{3}}{2}|T_0|$ and l_1 is on the left side of l as one faces in the direction of T, hence $l_1 = T'(l)$ while $|T'| = \frac{\sqrt{3}}{2}|T_0|$ and $\measuredangle\left(l, \overrightarrow{T'}\right) = \frac{2\pi}{4}$. Let $P \in l_1$ any point on l_1, then $\mathrm{d}(P, f_0(O)) = \sqrt{3}|T_0|$ and $\mathrm{d}(f_0(P), f(P)) = |T_0|$. Therefore the triangle $\triangle P f_0(P) f(P)$ is a right triangle with hypotenuse $Pf(P)$ of length is $2|T_0| = |T|$ and at an angle of $\frac{2\pi}{3}$ from the direction of T along l. Therefore ST carries $f(P)$ to P, and $(^ST \circ f)(P) = P$ for every $P \in l_1$, whence $^ST \circ f = R_{l_1}$.

If $f_0 \in G \setminus G_+$ satisfies A1, then its axis is parallel to T_1, and the axis of $f_0 \circ S$ is parallel to T_2. The result above shows that G then contains reflections f_1, f_2 whose mirrors are parallel to the direction of T_1, T_2 acordingly. Their product gives $S_{P,\frac{2\pi}{3}}$ while P is the meeting point of the mirrors, and we have got **p3m1**.

If $f_0 \in G \setminus G_+$ satisfies A2, then its axis paralllel to $T_1 \circ T_2$, and the same argument shows again that G contains reflections f_1, f_2 whose mirrors are perpendicular to the direction of T_1, T_2, and their product gives $S_{P,\frac{2\pi}{3}}$, where P is the meeting point of the mirrors, and we have got **p31m**. ■

2.3.5 Case 6: There are Rotations through $\frac{2\pi}{6}$ in G_+: $G \neq G_+ = \left\langle S_{O,\frac{2\pi}{6}} \right\rangle \ltimes T(G)$

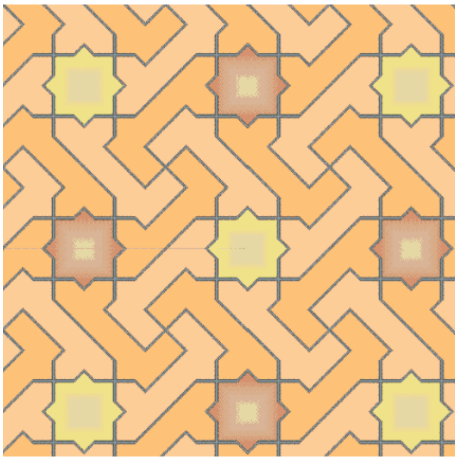
Proof: Let $S \triangleq S_{O,\frac{2\pi}{6}}$. Since $o(S^2) = 3$, we can apply the first argument from the previous proof again. There exists a reflection f whose mirror l is parallel to the direction of T_1, therefore the mirror l_1 of $f_1 = f \circ S$ satisfies $\measuredangle(l, l_1) = \frac{2\pi}{12}$. The product of the reflections give $S_{l \cdot l', \frac{2\pi}{6}}$. We have got **p6m**, where through every center P of order 6, there are 12 mirrors which meet at P: 6 reflections satisfy A1 and 6 reflections satisfy A2. ■

Finally, the proof of the Crystallographic Restriction Theorem has finished. Now, we are able to analize any drawing with a symmetry group G, which is a tessellation group as well, and to classify G to one of the 17 **tessellation types**.

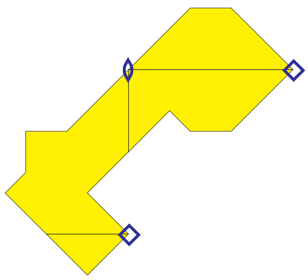
Summary

In the first chapter we have met lattices, tessellation groups, and their properties. We have seen the construction of a tile arising from the tessellation group, and we have checked the properties of a lattice symmetry-group as a tessellation one. In the second chapter we have seen a proof, whose ideas can be applied to other spaces as well.

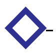
This summary is dedicated to one of the many applications of the crystallographic restriction theorem. We can now take a drawing, find its symmetries and construct its tessellation group with the standard tile. Let's look again at one of the drawings from the introduction:



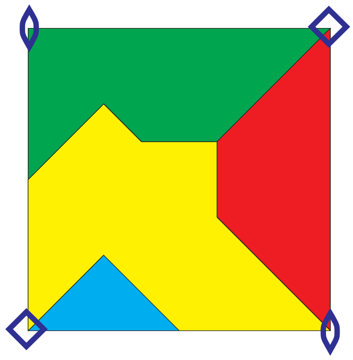
If the colouring has no meaning, it is trivial that the 'real' tile P is:



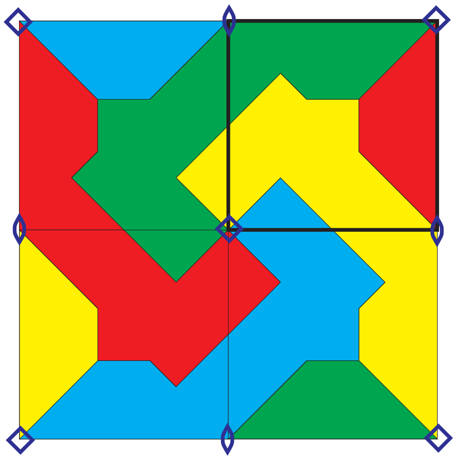
3 singular points of the tessellation group G are on the border of P, $\partial P = P \setminus \mathring{P}$, and are represented as the following figures:

 - the center of a half-turn; and - center of a rotation through $\frac{2\pi}{4}$.

If we cut P according to the added lines, we can build a square:



This square is a fundamental region for G, because P (with no symmetries in G) is a fundamental-region as well. Every $g \in G$ will transform the square onto another one. The ford-region in this case is the union of 4 squares:



We see a singular point O in the center of the ford-region, and 8 singular points on its border. The lattice $T(G) \cdot O$ is a square one, with symmetry-group $D_4 \ltimes T(G)$, where $T(G)$ is generated from two perpendicular translations T_1, T_2, and $|T_1| = |T_2|$ is twice the length of the square. It is $D_4 = \mathrm{Sym}(D(G;O))$ which tells us to cut $D(G;O)$ to 4 squares (and not 8 triangles, because $G = G_+$).
Here comes the crystallographic restriction theorem: our G can be only in one isometry-type which such lattice and fundamental-region - the **p4** isometry-type.
If the colouring is significant, then two of the previous tiles with different colours must be combined into the new tile $Q = P \cup H(P)$ where H is a half-turn whose center lies on the border of P. The same process is applied again, where the ford-region is the same, but it contains two fundamental-regions: both are rectangles that are combined from two colouring-distinct squares (the previous fundamental-regions). This type of tessellation is of type **p2**.
To finish, it is worth mentioning that the crystallographic restriction theorem holds (with other isometry-types and other amount) in higher dimensions and other geometries. For example, in $\mathbb{E}^3$ there are 230 isometry-types, and most of them can be found in crystals[1]. On the other hand, there are only 7 possible patterns on a strip, and on a line - only 2. But this theorem does not (and cannot) cover all patterns and drawings, and an example is **Penrous Chickens** (see figure 3): this drawing is a repeating pattern of two chickens (two different tiles), that can fill the whole plane, and this pattern does not have any symmetries at all, hence there is no tessellation group available.

[1] hence the name of the theorem ...

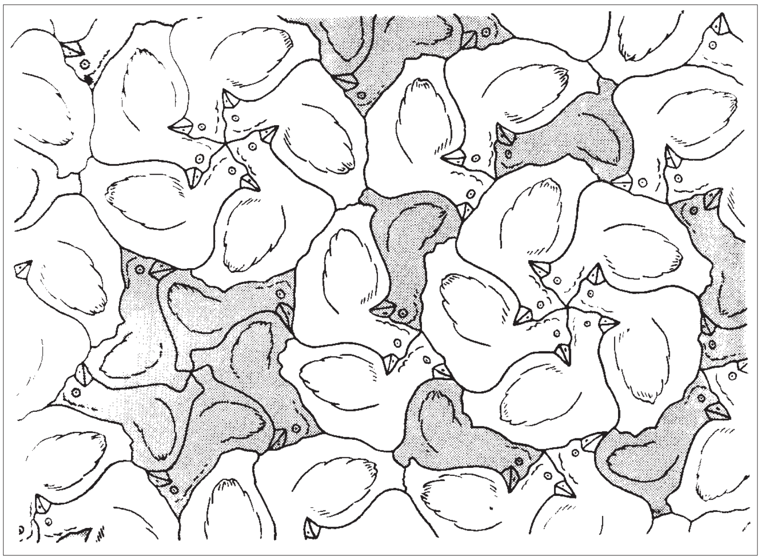


Figure 3: Penrous Chickens is a pattern without a tessellation group

Bibliography

[B] Berger M. (1984), "Geometry" (Russian), "MIR" (Moscow).

[C] Coxeter H.S.M., (1969), "Introduction to Geometry" (2^{nd} Edition), John Wiley & Sons (Canada), Inc.

[H] Herstein I.N. (1975), "Topics in Algebra" (2^{nd} Edition), John Wiley & Sons (USA), Inc.

[L] Lyndon Roger C. (1985), "Groups and Geometry", Cambridge University Press (London).